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Technical Report 32-1061

Nonsteady Evaporation of Liquid Propellant Drops: The Grossman Model

J. G. Sotter

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JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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Approved by:

D. F. Dipprey, *Manager*
Liquid Propulsion Section

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Abstract

Grossman's analysis of heat transfer to a liquid propellant droplet during resonant combustion is explained in a step-by-step derivation. For mathematical simplicity, Grossman assumes spherical symmetry; and radial convection within the drop is assumed to be negligible, at least near the surface. Furthermore, the evaporation history is arbitrarily divided into two parts: (1) an initial heatup period in which vaporization rate is set equal to zero, and heat is found to be conducted to a thin layer at the surface, and (2) the period of evaporation, in which the balance must be found between heat conducted to the drop and heat carried away by the vapors. In analyzing the latter period, Grossman assumes a constant surface temperature and requires that Fourier's Law of Heat Conduction need only be *approximately* satisfied. In this way, he avoids determining the local rates of temperature rise within the drop, and obtains the time rate of change of vaporization rate as a nondimensional expression which can be easily integrated numerically. The results seem to show that there is not much time for conduction in the drop, so that most of the incoming heat goes into evaporation.

Nonsteady Evaporation of Liquid Propellant Drops: The Grossman Model

I. Background of the Study

A. Introduction

Combustion in rocket engines is sometimes accompanied by undesirable high-amplitude, steep-fronted waves. When these waves are periodic, with a frequency over a few hundred cycles per second, the phenomenon is sometimes termed "resonant combustion." The energy to sustain the waves presumably is supplied by the high-frequency response of the burning process to time-varying local conditions.

To aid in the analytical portion of an investigation of resonant combustion, the Jet Propulsion Laboratory engaged the services of V. D. Agosta and S. Z. Burstein as consultants. Some related work was produced as a Master's thesis by Bernard Grossman, who at the time was one of Prof. Agosta's students at the Polytechnic Institute of Brooklyn, and this work was included for information in one of the consultant's reports to JPL. Grossman's work, which is based on an ablation study by R. F. Parisse, concerns the time-dependent heat transfer in an evaporating propellant droplet, and may be helpful in understanding the processes controlling evaporation rate under the rapid variations in heat transfer rate which are a part of the resonant combustion phenomenon. In many previous analyses, liquid propellant circulation within the drop was sufficiently rapid that the liquid temperature could be considered uniform throughout the drop. During rapid transients, however, this assumption is not necessarily justified. Grossman therefore investigates the consequences of making the opposite assumption: that there is *no* radial circulation within the drop. The validity of this assumption is also open to question; little is known about the subject in the case of very small drops. Surface effects would seem to discourage circulation; however, it is important to remember that large velocities are not necessary in order for circulation to be important. In a drop of 25-μ radius, a radial velocity component of only 10 ft/sec means that the fluid will travel one radius in less than 10 μsec; this could invalidate the conduction model. Nevertheless, in the absence of information on the effects of circulation, Grossman's analysis may be useful in establishing an "upper limit" for the response of the drop's evaporation rate to rapid changes in heat transfer rates.

During nearly three decades of research on resonant combustion, a great deal of analytical work has been produced. It is impossible for any one worker in the field to be familiar with the details of all the existing work, simply because of the time which would be involved. Yet, many engineers feel that to properly understand the work to the point of being able to apply it to real engines, they would like to understand each step in the analytical process; in short, to remove the appearance of "mystery" represented by mathematical manipulations which must be omitted from any concise presentation of the work. It is for these engineers that the present report is written. We do not attempt to be concise; brevity is sacrificed for what we hope is clarity. No original work (by the present writer) will be presented: rather, a step-by-step "expanded version" of Grossman's droplet evaporation study is given in Sec. II and III, which are reprinted with minor editorial modifications from sections of JPL internal memoranda. The expanded versions were written in an attempt to make the work easily understandable to engineers who would not have the time to work out the analysis for themselves. Only a background of elementary calculus is assumed; it is hoped that the derivations can be followed "by eye," and at least 90% of the steps can be confirmed by the reader without using pencil and paper. For a full understanding of the work, a careful reading followed by a review is recommended.

We do not pretend (nor did Grossman) that the evaporation analysis presented here is the ultimate in physical or mathematical rigor; rather, it is an attempt to reduce a very complex problem to an approximate solution which is simple enough to be of potential usefulness in the still *more* complex numerical calculations of the aerothermochemistry of resonant combustion.

B. Outline of the Analysis

The basic relation around which the analysis is built is the *heat-conduction equation.* In one-dimensional form, this equation can be written:[1]

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where T is temperature, t is time, x is spatial coordinate, and α is the thermal diffusivity of the material. This form of the equation is easily interpreted. Let us consider a a solid slab which is perfectly insulated ($\partial T/\partial x = 0$) on both faces and which at time zero has some arbitrary temperature distribution $T(x)$ such as the hypothetical one given by the solid line in Fig. 1. Instinct suggests that in the regions where the curve is *concave downward* (such as region A), the temperature will *decrease* as time passes, while in the regions where the curve is *concave upward* (as in region B), the temperature will *increase.* The heat-conduction equation states this feeling more explicitly by saying that the time rate of temperature rise at point x and time t is proportional to the degree of concavity of the T vs x curve; and that the proportionality factor is the thermal diffusivity α, which is the ratio of the thermal conductivity to the heat capacity per unit volume.

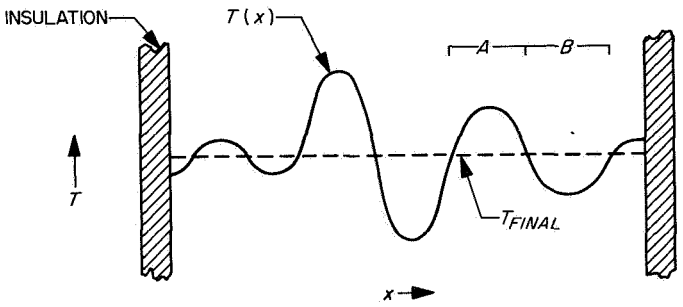

Fig. 1. Hypothetical, one-dimensional temperature distribution in a thick slab

In problems with spherical symmetry, the *form* of the equation is the same, but since energy must still be conserved, a simple substitution is made to account for the fact that the volume-to-surface area ratio of a thin spherical shell is proportional to radius. (This form of the equation will be derived in Section II.)

For mathematical simplicity, Grossman divides the droplet history into two distinct periods: (1) initial heatup and (2) evaporation. The *initial heatup* is defined to begin when a spherical droplet, at its injection temperature, is suddenly exposed to the combustion chamber environment. Surface reactions are ignored; convective heat flux to the liquid surface is obtained as a function of (1) the temperature difference across the gaseous "film" around the droplet and (2) the "film" heat-transfer coefficient, obtained from an empirical formula. As a simplification, it is assumed that evaporation is negligible during the initial heatup period. The heat-conduction equation is then solved for the spherically symmetric case with the aid of Laplace transforms; this method essentially reduces the problem: (1) to transforming to a frequency domain, (2) to solving an algebraic equation, and (3) to looking up the inverse transform in tables. The initial heatup solution is presented in Section II.

[1]The nomenclature is listed at the end of the report.

The results of this heatup analysis show that there is very little penetration of heat to the subsurface region, since only a few microseconds are required for the droplet surface to reach its "vaporization temperature." When this temperature is reached, *evaporation* is considered to begin, with all vapor being removed as soon as it is formed. Evaporation is assumed to occur at a known constant surface temperature; thus, for a rapid heat input, any heat which has not time to be conducted to the interior of the drop must be used for evaporation of the liquid at the surface. A step discontinuity with exponential decay is assumed for the time history of the heat input, and the subsequent temperature profile within the drop is approximated by an exponential function. By requiring that Fourier's law of heat conduction need only be *approximately* satisfied, Grossman avoids determining the local rates of temperature rise within the drop, and obtains the time rate of change of vaporization rate as a nondimensional expression which can easily be integrated numerically. For the cases calculated, the analysis seems to show that there is little time for conduction, and most of the heat does in fact go into evaporation. The evaporation analysis is given in Section III.

Grossman's analysis may be considered to be an "order-of-magnitude" first approximation to the problem of heat transfer within a drop for small times. The results indicate that if radial convection near the drop's surface is negligible, then the assumption of uniform droplet temperature, which is used in many liquid propellant combustion theories, may be seriously in error. The conclusion is that further refinement of the model is justified.

II. Droplet Heatup in the Absence of Vaporization[2]

A. Droplet Heating

The fundamental hypothesis for the mathematical theory of heat conduction is that the rate at which heat crosses from the inside to the outside of an isothermal surface per unit area per unit time is equal to

$$-k\frac{\partial T}{\partial n}$$

where T is temperature, k is the thermal conductivity of the substance, and $\partial/\partial n$ denotes differentiation along the outward-drawn normal to the surface. Using this hypothesis and considering the flow of heat across the six surfaces of a rectangular parallelepiped, one can derive *the basic equation for conductive heat transfer* in rectangular coordinates (p. 9 of Ref. 1):

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{1}$$

where t is time, and α is the thermal diffusivity, related to conductivity by $\alpha = k/\rho c$, ρ being the density and c the specific heat.

Expressing Eq. (1) in spherical coordinates (p. 17 of Ref. 1) and assuming spherical symmetry, yields (p. 230 of Ref. 1)

$$\boxed{\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right)} \tag{2}$$

where r is the radial coordinate.

The droplet heat transfer model is given in Fig. 2. The heat transferred into the drop is expressed in terms of

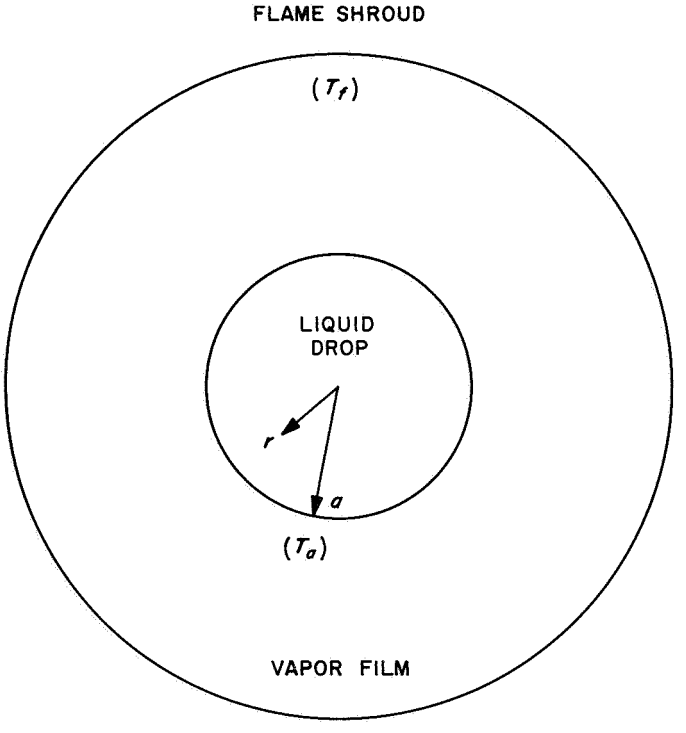

Fig. 2. Combustion model

[2]This section is based on *Aerothermochemical Analysis of Non-oscillatory on Oscillatory Characteristics of Liquid Bipropellant Rocket Motors: Wave Model, Droplet Heating, and Evaporation Program Computation*, by V. D. Agosta and S. Z. Burstein, Report 65-1, Third Quarterly Report, January 15, 1965 (Prepared for Jet Propulsion Laboratory) (STAR N66-13083).

the temperature difference across a spherical "film" having a coefficient of heat transfer h:

$$h(T_f - T_a) = k\frac{\partial T}{\partial r}\ ;\quad r = a, t \geq 0 \tag{3a}$$

where a is the initial drop radius, T_a is surface temperature, and T_f is the stagnation temperature of the flow field. At the center of the droplet, giving T as a function of coordinates (r,t) we have

$$T(0,t) \text{ is finite} \tag{3b}$$

and the initial condition is a uniform droplet temperature T_i: T at all r is equal to T_i when $t = 0$.

$$T(r,0) = T_i \tag{3c}$$

The discussion in the following four paragraphs is subsidiary to the main argument. A solution to this equation is given by H. S. Carslaw and J. C. Jaeger, in *Conduction of Heat in Solids*, Oxford at the Clarendon Press, 1950, p. 234, as

$$\frac{T_f - T}{T_f - T_i} = \frac{2h}{kr}\sum_{n=1}^{\infty}\frac{(a^2\beta_n^2+\mu^2)\sin(\beta_n a)\sin(\beta_n r)}{\beta_n^2\left[a^2\beta_n^2+\mu(\mu+1)\right]}\rho^{-\beta_n^2\alpha t} \tag{4}$$

where β_n is given by the solution of

$$\mu + \beta_n a\cot(\beta_n a) = 0 \tag{5}$$

(tabulated in Ref. 1) and

$$\mu \equiv \left(\frac{ah}{k} - 1\right) \tag{6}$$

"For 50/50 hydrazine-UDMH blend at about 650°R[3]:

$\rho \simeq 53$ lbm/ft^3

$C_p \simeq 0.694$ Btu/lbm°R

$k \simeq 0.42 \times 10^{-4}$ Btu/sec ft°R

$\alpha \simeq 1.15 \times 10^{-6}$ ft^2/sec

"For O/F$\cong$2.20, the flame temperature near the injector end of the chamber is $T_f \simeq 5800$°R and $h = 1.5$ Btu/ft^2 sec°R (using Ref. 2) to a droplet with $a \simeq 0.375 \times 10^{-3}$ ft (~ 0.004 in.).

"Reference is now made to Eq. (4). For short time intervals [related to the stay time of a droplet in a rocket engine], i.e., $10^{-6} \leq t \leq 10^{-3}$, $e^{-\beta_n^2\alpha t} \simeq e^0 = 1$. In addition, $\sin\beta_n a \leq 1$ and $\sin\beta_n r \leq 1$. Therefore, the rate of convergence of this solution for short time intervals is dependent on the factor,"

$$\frac{2h}{kr}\sum_{n=1}^{\infty}\frac{(a^2\beta_n^2+\mu^2)}{\beta_n^2\left[a^2\beta_n^2+\mu(\mu+1)\right]}$$

Unfortunately, for small times, this solution converges very slowly, making it of no practical use in the time range of interest. Thus, a short term solution is sought.

The method of solution again is found in Carslaw and Jaeger (Ref. 3) and was used by Parisse (Ref. 4). The heating up solution is reproduced below, and modified to the extent required to make the terms apply to a liquid fuel droplet. [Again without circulation.]

"A short-time solution for the temperature distribution in a liquid sphere subjected to a point symmetric heat input is obtained by utilizing Laplace transformations."

A first step in this solution is to transform Eq. (2) into a more convenient form by defining a parameter u such that

$$u \equiv r(T_f - T) \tag{7}$$

which may be rearranged to give

$$T = T_f - \frac{u}{r}$$

so that

$$\frac{\partial T}{\partial t} = -\frac{1}{r}\frac{\partial u}{\partial t} \tag{8}$$

[3] Material in quotation marks is taken from B. Grossman's Master's thesis, as supplied by Prof. Agosta; changes have been made to ensure consistency with the present report in such matters as reference and equation numbers. Generally, material in brackets has been supplied by J. G. Sotter to add continuity to the presentation.

taking T_f to be constant. Also,

$$\frac{\partial T}{\partial r} = ur^{-2} - r^{-1}\frac{\partial u}{\partial r} \tag{9}$$

$$\frac{\partial^2 T}{\partial r^2} = -2\,ur^{-3} + r^{-2}\frac{\partial u}{\partial r} - r^{-1}\frac{\partial^2 u}{\partial r^2} + r^{-2}\frac{\partial u}{\partial r} \tag{10}$$

Now we can substitute Eq. (8), (9), and (10) into Eq. (2), giving

$$-r^{-1}\frac{\partial u}{\partial t} = \alpha\left[-2\,ur^{-3} + r^{-2}\frac{\partial u}{\partial r} - r^{-1}\frac{\partial^2 u}{\partial r^2} + r^{-2}\frac{\partial u}{\partial r} + \frac{2}{r}\left(ur^{-2} - r^{-1}\frac{\partial u}{\partial r}\right)\right]$$

If we multiply by r and remove the brackets, most of these terms cancel each other, leaving

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial r^2}, \qquad 0 \le r < a \tag{11}$$

as the new form for Eq. (2). To get the corresponding hot boundary condition, we substitute into Eq. (3a),

$$h\,(T_f - T_a) = k\frac{\partial T}{\partial r}, \qquad r = a \tag{3a}$$

using Eq. (7) and (9) and $r = a$. This gives

$$h\frac{u}{a} = k\left(ua^{-2} - a^{-1}\frac{\partial u}{\partial r}\right)$$

Now we multiply both sides by a/k and remove parentheses:

$$\frac{hu}{k} = \frac{u}{a} - \frac{\partial u}{\partial r}$$

or

$$\left(\frac{hu}{k} - \frac{u}{a}\right) + \frac{\partial u}{\partial r} = 0$$

Now factor out u/a from the term in parentheses:

$$\frac{u}{a}\left(\frac{ah}{k} - 1\right) + \frac{\partial u}{\partial r} = 0$$

And remembering the definition of μ from Eq. (6),

$$\frac{\partial u}{\partial r} + \frac{\mu}{a}u = 0, \qquad r = a \tag{12a}$$

becomes the new form of Eq. (3a) in terms of u. For Eq. (3b) and (3c), using Eq. (7) directly, we write

$$u\,(r, 0) = r\,(T_f - T_i) \tag{12b}$$

$$u\,(0, t) = 0 \tag{12c}$$

We are now ready to solve Eq. (11). This will be accomplished by taking its "Laplace transform," and integrating the resulting "subsidiary equation," which incorporates the initial condition (Eq. 12b); the definite integral will then be transformed back to the original frame of reference.

Restating Eq. (11),

$$\frac{\partial^2 u}{\partial r^2} - \frac{1}{\alpha}\frac{\partial u}{\partial t} = 0 \tag{13}$$

we apply the Laplace transformation; that is, we multiply both sides by e^{-pt} and integrate with respect to t from 0 to ∞.

$$\int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial r^2}\,dt - \frac{1}{\alpha}\int_0^\infty e^{-pt}\frac{\partial u}{\partial t}\,dt = 0 \tag{14}$$

where p is a number whose real part is positive and large enough to make the integrals convergent. Equation (14) is more conveniently written by defining the "Laplace transform,"

$$L\{u\} \equiv \bar{u} \equiv \int_0^\infty e^{-pt}\,u\,dt$$

Since r is independent of t, Eq. (14) is not changed by moving $(\partial^2/\partial r^2)$ outside the integral sign; Eq. (13) or (14) becomes

$$\frac{\partial^2 \bar{u}}{\partial r^2} - \frac{1}{\alpha}L\left\{\frac{\partial u}{\partial t}\right\} = 0$$

Using another basic property of Laplace transforms (see Ref. 1) (obtained simply from integration by parts in the

second term), we introduce the initial condition.

$$\frac{\partial^2 \bar{u}}{\partial r^2} - \frac{1}{\alpha}\left[p\bar{u} - u(r,0)\right] = 0$$

which, using Eq. (12b), becomes

$$\frac{\partial^2 \bar{u}}{\partial r^2} - \frac{1}{\alpha}\left[p\bar{u} - r(T_f - T_i)\right] = 0$$

giving the so-called "subsidiary equation,"

$$\frac{\partial^2 \bar{u}}{\partial r^2} - \frac{p}{\alpha}\,\bar{u} = -\frac{r\,(T_f - T_i)}{\alpha}$$

Since no differentiation with respect to p is involved, we may treat the partial as a total derivation. Defining $q^2 \equiv p/\alpha$,

$$\frac{d^2 \bar{u}}{dr^2} - q^2\bar{u} = -\frac{r\,(T_f - T_i)}{\alpha} \tag{15}$$

This is the equation to be integrated. The integral can be expressed (see Ref. 6) as the sum of a particular solution $\bar{u}_p$, and the complementary function $\bar{u}_c$. The latter is found by integrating the homogeneous equation formed by setting the right-hand side of Eq. (15) equal to zero:

$$\frac{d^2 \bar{u}}{dr^2} - q^2\bar{u} = 0$$

which, expressed in operator form, is

$$(D^2 - q^2)\,\bar{u} = 0$$

$$(D + q)\,(D - q)\,\bar{u} = 0$$

where

$$D \equiv \frac{d}{dr}$$

Now first let

$$(D - q)\,\bar{u} = 0 \tag{16}$$

or

$$\frac{d\bar{u}}{dr} = q\,\bar{u}$$

$$\frac{d\bar{u}}{\bar{u}} = q\,dr$$

Integrating,

$$\ln \bar{u} = qr + c$$

$$\bar{u} = e^{qr+c}$$

$$= C_1\,e^{qr} \tag{17}$$

where c and C_1 are constants. Similarly, letting

$$(D + q)\,\bar{u} = 0$$

and repeating the steps from Eq. (16) to (17), we obtain a second solution,

$$\bar{u} = C_2\,e^{-qr}$$

so that the complementary equation is

$$\bar{u}_c = C_1\,e^{qr} + C_2\,e^{-qr}$$

Now to find a *particular* integral of Eq. (15). We can assume (see Ref. 6) that such a solution may be written as

$$\bar{u}_p = k_1 r + k_2 \tag{18}$$

that is, as a polynomial of the same order as the right-hand side of Eq. (15); k_1 and k_2 are constants, to be determined. Obviously, from Eq. (18),

$$\frac{d^2 \bar{u}_p}{dr^2} = 0$$

Substituting into Eq. (15) yields

$$-(q^2 k_1)\,r - q^2 k_2 = -\frac{(T_f - T_i)\,r}{\alpha}$$

Since the two sides are identically equal, coefficients of like powers of r can be equated. This gives

$$k_1 = \frac{T_f - T_i}{q^2\alpha}$$

$$k_2 = 0$$

so Eq. (18) becomes

$$\bar{u}_p = \frac{(T_f - T_i)}{q^2\alpha} r$$

and the general solution of Eq. (15) is

$$\bar{u} = \bar{u}_c + \bar{u}_p = C_1 e^{qr} + C_2 e^{-qr} + \frac{(T_f - T_i)\, r}{q^2\alpha} \tag{19}$$

Now we evaluate the constants. First we must express the boundary conditions in the transform domain. Taking the "transform" of the constant, zero, Eq. (12c) becomes

$$\bar{u} = 0 \qquad \text{at } r = 0 \tag{20c}$$

Using this, we quickly see that $C_1 = -C_2$. The other boundary condition, (Eq. 12a), can be expressed using a total derivative. It becomes

$$\frac{d\bar{u}}{dr} + \frac{\mu}{a}\bar{u} = 0 \qquad \text{at } r = a \tag{20a}$$

Equation (19) is easily differentiated as we substitute in the above, using $C_2 = -C_1$:

$$C_1 q\,(e^{qa} + e^{-qa}) + \frac{T_f - T_i}{q^2\alpha} + \frac{\mu}{a}\left[C_1 e^{qa} - C_1 e^{-qa} + \frac{(T_f - T_i)a}{q^2\alpha}\right] = 0$$

Rearranging, and remembering $q^2 \equiv p/\alpha$,

$$C_1\left[q\,(e^{qa} + e^{-qa}) + \frac{\mu}{a}(e^{qa} - e^{-qa})\right] + \frac{T_f - T_i}{p}(1 + \mu) = 0$$

$$C_1 = -\frac{(T_f - T_i)(1 + \mu)}{p\left[\left(q + \frac{\mu}{a}\right)e^{qa} + \left(q - \frac{\mu}{a}\right)e^{-qa}\right]}$$

Now we return to Eq. (19), inserting the constants. First use $C_2 = -C_1$

$$\bar{u} = C_1(e^{qr} - e^{-qr}) + \frac{(T_f - T_i)\, r}{p} = -\frac{(T_f - T_i)(1 + \mu)(e^{qr} - e^{-qr})}{p\left[\left(q + \frac{\mu}{a}\right)e^{qa} + \left(q - \frac{\mu}{a}\right)e^{-qa}\right]} + \frac{(T_f - T_i)\, r}{p}$$

Rearranging,

$$\bar{u} = \frac{(T_f - T_i)\, r}{p} - \frac{(1 + \mu)(T_f - T_i)}{p}\left[\frac{e^{qr} - e^{-qr}}{\left(q + \frac{\mu}{a}\right)e^{qa} + \left(q - \frac{\mu}{a}\right)e^{-qa}}\right] \tag{21}$$

The term in brackets gives an infinite series by long division. We start with e^{qr}:

$$\left(q+\frac{\mu}{a}\right)^{-1}e^{-qa+qr}-\frac{\left(q-\frac{\mu}{a}\right)}{\left(q+\frac{\mu}{a}\right)^{2}}e^{-3qa+qr}+\frac{\left(q-\frac{\mu}{a}\right)^{2}}{\left(q+\frac{\mu}{a}\right)^{3}}e^{-5qa+qr}-\frac{\left(q-\frac{\mu}{a}\right)^{3}}{\left(q+\frac{\mu}{a}\right)^{4}}e^{-7qa+qr}\cdots$$

$$\left(q+\frac{\mu}{a}\right)e^{qa}+\left(q-\frac{\mu}{a}\right)e^{-qa}\;\Big)\overline{e^{qr}}$$

$$e^{qr}+\frac{\left(q-\frac{\mu}{a}\right)}{\left(q+\frac{\mu}{a}\right)}e^{-2qa+qr}$$

$$-\frac{\left(q-\frac{\mu}{a}\right)}{\left(q+\frac{\mu}{a}\right)}e^{-2qa+qr}$$

$$-\frac{\left(q-\frac{\mu}{a}\right)}{\left(q+\frac{\mu}{a}\right)}e^{-2qa+qr}-\frac{\left(q-\frac{\mu}{a}\right)^{2}}{\left(q+\frac{\mu}{a}\right)^{2}}e^{-4qa+qr}$$

$$+\frac{\left(q-\frac{\mu}{a}\right)^{2}}{\left(q+\frac{\mu}{a}\right)^{2}}e^{-4qa+qr}$$

$$+\frac{\left(q-\frac{\mu}{a}\right)^{2}}{\left(q+\frac{\mu}{a}\right)^{2}}e^{-4qa+qr}+\frac{\left(q-\frac{\mu}{a}\right)^{3}}{\left(q+\frac{\mu}{a}\right)^{3}}e^{-6qa+qr}$$

$$-\frac{\left(q-\frac{\mu}{a}\right)^{3}}{\left(q+\frac{\mu}{a}\right)^{3}}e^{-6qa+qr}$$

or

$$\frac{e^{qr}}{\left(q+\frac{\mu}{a}\right)e^{qa}+\left(q-\frac{\mu}{a}\right)e^{-qa}}=\sum_{n=0}^{\infty}(-1)^{n}\frac{\left(q-\frac{\mu}{a}\right)^{n}}{\left(q+\frac{\mu}{a}\right)^{n+1}}e^{-q[(2n+1)a-r]}$$

Adding to this a similar series for $-e^{-qr}$, we obtain the series form of Eq. (21),

$$\bar{u}=\frac{(T_f-T_i)\,r}{p}-\frac{(1+\mu)\,(T_f-T_i)}{p}\sum_{n=0}^{\infty}(-1)^{n}\frac{\left(q-\frac{\mu}{a}\right)^{n}}{\left(q+\frac{\mu}{a}\right)^{n+1}}\left\{e^{-q[(2n+1)a-r]}-e^{-q[(2n+1)a+r]}\right\}\tag{22}$$

"For a short-time solution, Eq. (22) can be truncated after the $n = 0$ term. For the range of parameters considered in this report, the higher order terms of Eq. (22) have been shown to be negligible." [Reference 5 gives inverse transforms for higher order terms.] Using this approximation (which has not been rigorously proven) we have

$$\bar{u} \doteq \frac{(T_f - T_i)\, r}{p} - \frac{(1+\mu)\,(T_f - T_i)}{p} \left\{ \frac{1}{q + \frac{\mu}{a}} \left[e^{-q(a-r)} - e^{-q(a+r)} \right] \right\}$$

We now refer this solution back to the original domain. Consulting a table of Laplace transforms (p. 494, No. 1 and 14 of Ref. 1), we find

$$u \doteq (T_f - T_i)\, r - (1+\mu)\,(T_f - T_i) \left\{ \frac{a}{\mu}\, erfc \frac{a-r}{2\sqrt{\alpha t}} - \frac{a}{\mu} \exp\left[\frac{\mu}{a}(a-r) + \alpha t \frac{\mu^2}{a^2} \right] \left(erfc \frac{a-r}{2\sqrt{\alpha t}} + \frac{\mu}{a}\sqrt{\alpha t} \right) \right.$$
$$\left. + \frac{a}{\mu}\, erfc \frac{a+r}{2\sqrt{\alpha t}} - \frac{a}{\mu} \exp\left[\frac{\mu}{a}(a+r) + \alpha t \frac{\mu^2}{a^2} \right] \left(erfc \frac{a+r}{2\sqrt{\alpha t}} + \frac{\mu}{a}\sqrt{\alpha t} \right) \right\}$$

Where the "error complementary function" *erfc X* is defined by

$$erfc\, X \equiv \frac{2}{\sqrt{\pi}} \int_X^\infty e^{-\xi^2}\, d\xi \qquad (23)$$

(Tables of this function are readily available, although care must be taken that the correct definition is used.)

Introducing the quantities

$$\tau = \frac{\alpha t}{a^2}$$

$$r' = r/a$$

and recalling the definition of u from Eq. (7), we obtain a nondimensional temperature

$$\frac{T - T_i}{T_f - T_i} = \frac{1}{r'} \left[\frac{1+\mu}{\mu} \right] \left\{ erfc \frac{1-r'}{2\sqrt{\tau}} - erfc \frac{1+r'}{2\sqrt{\tau}} \right.$$
$$- exp\,[\mu(1-r') + \mu^2 \tau]\, erfc \left[\frac{1-r'}{2\sqrt{\tau}} + \mu\sqrt{\tau} \right]$$
$$\left. + exp\,[\mu(1+r') + \mu^2 \tau]\, erfc \left[\frac{1+r'}{2\sqrt{\tau}} + \mu\sqrt{\tau} \right] \right\} \qquad (24)$$

This is the required solution to Eq. (2) subject to the conditions of Eq. (3a), (3b), and (3c).

"A plot was made of the heating up of a fuel droplet, (50/50 N_2H_4/UDMH, for $a = 0.375 \times 10^{-3}$ ft) and is shown in Fig. 3. It is seen that in a very short time the surface of the drop heats up." At 300 psi, the boiling point (see Ref. 7) of this fuel is 860°R (400°F). Assuming

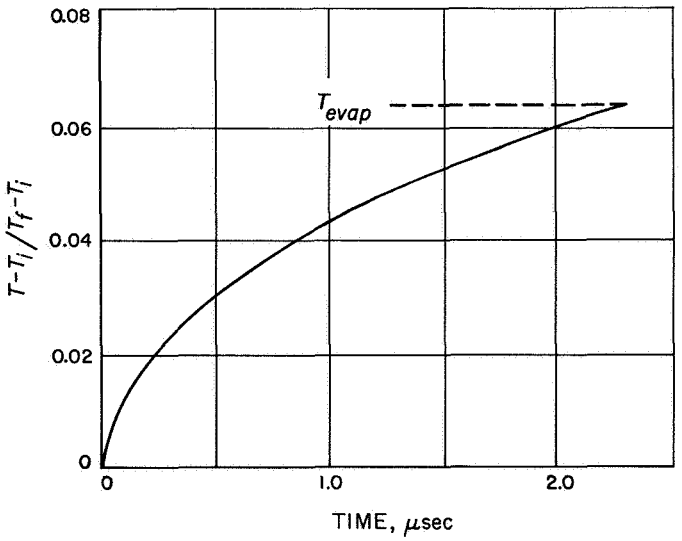

Fig. 3. Heating of droplet surface

an initial temperature of 528°R (68°F), and using T_f = 5800°R, we see that at the boiling point the nondimensional temperature given by Eq. (24) will be

$$\frac{T_{evap} - T_i}{T_f - T_i} = \frac{860 - 528}{5800 - 528} = \frac{332}{5272} = 0.063$$

Using data given on p. 4, and assuming that vaporization begins when the boiling point is reached, Fig. 3 shows that the heatup time before evaporation occurs is about 2.3 μsec.[4]

In order to represent temperature distribution *inside* the droplet, we shall use a different nondimensional temperature

$$\frac{T - T_i}{T_{evap} - T_i} = \left(\frac{T - T_i}{T_f - T_i}\right) \frac{T_f - T_i}{T_{evap} - T_i}$$

$$= \left(\frac{T - T_i}{T_f - T_i}\right) \times \frac{1}{0.063}$$

where the quantity in parentheses is given by Eq. (24).

"The temperature distribution in the droplet was determined for the case where the surface temperature was equal to the evaporation temperature. It is seen in Fig. 4 that the heating effect is concentrated in the outer 1% of the droplet radius. This agrees with results from the previous aerothermochemical analysis;" namely, that very little of the sensible heat absorbed by the drop is utilized in increasing the bulk temperature of the drop.

[4]Different data were used for Fig. 3 in the original Agosta and Burstein report, but results were qualitatively the same.

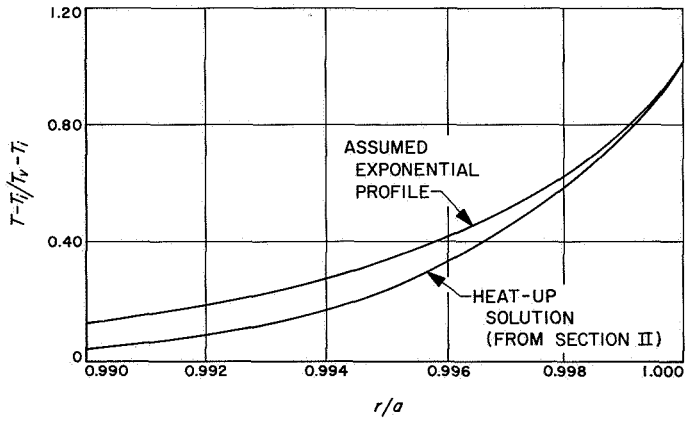

Fig. 4. Temperature distribution in droplet

Only the outer 1% of the radius is shown in Fig. 4; at smaller radii, the quantity

$$\frac{T - T_i}{T_{evap} - T_i}$$

is very close to zero.

"The next step in this analysis is to determine the evaporation rate for the droplet." This analysis is presented in Section III.

III. Droplet Vaporization: Heat-Transfer Model[5]

A. A Mathematical Note

Leibnitz's rule for differentiation under the integral sign states (with certain restrictions) that for a function

$$\phi(t) = \int_{r_1}^{r_2} f(r, t)\, dr$$

where r_1 and r_2 may depend on t, the derivative of ϕ is given by

$$\frac{d\phi}{dt} = \int_{r_1}^{r_2} \frac{\partial f}{\partial t}\, dr + f(r_2, t)\, \frac{dr_2}{dt} - f(r_1, t)\, \frac{dr_1}{dt}$$

The formula is illustrated by Fig. 5, in which the integral is represented by the area under the curve $f(t)$.

Combining both equations, and transposing two terms, we obtain

$$\int_{r_1}^{r_2} \frac{\partial f}{\partial t}\, dr = \frac{d}{dt} \int_{r_1}^{r_2} f(r, t)\, dr - f(r_2, t)\, \frac{dr_2}{dt} + f(r_1, t)\, \frac{dr_1}{dt}$$

Thus, if we can determine the values of the last two terms, Leibnitz's rule provides a means of trading a partial for a total derivative.

[5]This section is based on Part I of *Aerothermochemical Analysis of Non-oscillatory and Oscillatory Characteristics of Liquid Bipropellant Rocket Motors: Droplet Evaporation, Wave Equations, Motor Instrumentation, Non-oscillatory Program Computation,* by Vito D. Agosta, Report 65-2, April 15, 1965 (Prepared for Jet Propulsion Laboratory) (STAR N66-13634).

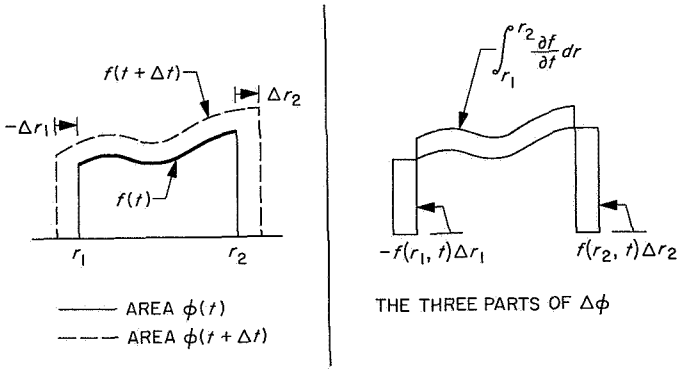

Fig. 5. Illustration of Leibnitz's rule

B. Evaporation Analysis

"As the temperature of the droplet increases, the mass evaporation becomes substantial and must be explicitly included in the boundary conditions of the vaporization solution." The governing equation for conduction within the drop is still assumed to be the spherically symmetric relation given by Eq. (2) in Section II.

The total heat transfer q_t from the surrounding gases goes to heating the liquid droplet q_l, vaporizing the surface liquid q_v, and heating the ensuing gases to the flame temperature, q_h.

$$q_t = q_l + q_v + q_h \tag{25}$$

"Choosing a convective model for the energy transport from the flame to droplet, Eq. (25) becomes

$$h(T_f - T_v) = k\left(\frac{\partial T}{\partial r}\right)_{r_s,t} + \frac{w}{A}\lambda + \frac{w}{A}c_p(T_f - T_v) \tag{26}$$

where λ is the heat vaporization, w is the mass evaporation rate, A is the surface area of the droplet, and $r_s(t)$ is the coordinate which locates the position of the outer surface at any time t."

At this point we wish to increase the generality of the heat flux expression by arbitrarily adding a term which might simulate the additional heat input caused by a shock wave. We use a step function which decays exponentially with time, so that Eq. (26) becomes

$$\bar{q}\, e^{-\frac{4t}{t^*}} + h(T_f - T_v) = k\left(\frac{\partial T}{\partial r}\right)_{r_s,t} + \frac{w}{A}\lambda + \frac{w}{A}c_p(T_f - T_v) \tag{27}$$

where $\bar{q}$ is the peak additional heat input, and t^* is the "decay time" of the wave.

"The conservation of mass equation in differential form is

$$w = -4\pi r_s^2 \rho \frac{dr_s}{dt} = -A\rho\frac{dr_s}{dt} \tag{28}$$

where ρ is the density of the liquid. Employing this result Eq. (27) becomes,"

$$k\left(\frac{\partial T}{\partial r}\right)_{r_s,t} = \bar{q}\, e^{-\frac{4t}{t^*}} + h(T_f - T_v) + \rho[\lambda + c_p(T_f - T_v)]\frac{dr_s}{dt} \tag{27a}$$

To save space, we shall use the symbols

$$Q \equiv \bar{q}\, e^{-\frac{4t}{t^*}} + h\,(T_f - T_v) \tag{27b}$$

$$L \equiv \lambda + c_p\,(T_f - T_v) \tag{27c}$$

So that Eq. (27a) becomes

$$k\left(\frac{\partial T}{\partial r}\right)_{r_s,t} = Q + \rho L\frac{dr_s}{dt} \tag{27d}$$

Note that if the steady-state heat input $h(T_f - T_v)$ is considered to vary only slowly as the drop moves down the chamber, then time-differentiation of Eq. (27b) gives simply

$$\frac{dQ}{dt} \doteq -\frac{4\bar{q}}{t^*}\, e^{-\frac{4t}{t^*}} \tag{27e}$$

"For the vaporization process, the heat conduction equation applies for the interval $0 \leq r \leq r_s(t)$. [Convection within the drop is ignored.] One boundary condition at the surface of the droplet is

$$\boxed{T(r_s, t) = T_v} \tag{29}$$

Although the vaporization temperature, T_v, will vary with time, its appearance in the following analysis will be as the difference $T_f - T_v$ which, within the approximations made in this analysis, remains constant. Now the

time variable is defined to be zero when $r_s = a$. Hence from a consideration of the continuity of heat input for the heatup and vaporization analysis

$$\frac{dr_s}{dt} = 0 \qquad \text{at } t = 0 \tag{30}$$

"The initial condition determined by the heatup analysis is some function

$$T(r, 0) = T_0(r) \tag{31}$$

"The system of Eq. (27a), (29), (30), and (31) [together with Eq. (2) in Section II] is nonlinear due to the presence of $r_s(t)$. Parisse (Ref. 4) outlined a method of applying Goodman's heat balance integral to the ablation of a solid sphere. As a result of the assumptions made in this analysis, the method of solution of the problem is similar."

Equation (1) will be converted to a more convenient form by the substitution $u = rT$. First we differentiate the latter.

$$T = \frac{u}{r} \tag{32}$$

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial u}{\partial t} \tag{33}$$

Equation (32) also leads to

$$\frac{\partial T}{\partial r} = -\frac{u}{r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \tag{34}$$

$$\frac{\partial^2 T}{\partial r^2} = \left(\frac{2u}{r^3} - \frac{1}{r^2}\frac{\partial u}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\frac{\partial u}{\partial r}\right) \tag{35}$$

Now we can substitute Eq. (33), (34), and (35) into the spherical heat conduction relation [Eq. (2) of Section II]. This gives

$$\frac{1}{r}\frac{\partial u}{\partial t} = \alpha\left(\frac{2u}{r^3} - \frac{2}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial r^2} - \frac{2u}{r^3} + \frac{2}{r^2}\frac{\partial u}{\partial r}\right)$$

Note that most of these terms cancel each other, leaving

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial r^2} \tag{36}$$

as the new form of the heat conduction equation. Mathematically, the equality is of course not affected if we integrate over the volume of the liquid:

$$\int_0^{r_s} \frac{\partial u}{\partial t}\, dr = \alpha \int_0^{r_s} \frac{\partial^2 u}{\partial r^2}\, dr \tag{37}$$

However, this step has physical consequences. Since we shall be solving the integrated equation (Eq. 37), we cannot expect that the proportionality between the two derivatives, given by the heat condition equation (Eq. 36), will be preserved at every point. We can only require that Eq. (36) be satisfied *on the average*; that is, that the *areas* under the two curves shall be proportional as stated in Eq. (37). Conservation of energy is still satisfied by Eq. (37); thus, it is referred to as the "heat-balance integral." Since our boundary conditions are specified, we can hope that the approximation implied by Eq. (37) will not introduce serious error.

Now we perform the integration in the right-hand side of Eq. (37) and use Leibnitz' rule (p.10) on the left-hand side, remembering that $u(0, t) = 0$.

$$\frac{d}{dt}\int_0^{r_s} u\, dr - u(r_s, t)\frac{dr_s}{dt} = \alpha\left[\left(\frac{\partial u}{\partial r}\right)_{r_s, t} - \left(\frac{\partial u}{\partial r}\right)_{0, t}\right]$$

Since $u = rT$, we have

$$\frac{d}{dt}\int_0^{r_s} Tr\, dr - r_s T(r_s, t)\frac{dr_s}{dt} =$$

$$\alpha\left[r_s\left(\frac{\partial T}{\partial r}\right)_{r_s, t} + T(r_s, t) - T(0, t)\right] \tag{38}$$

1. Assumed temperature profile. On the basis of the results of the initial heatup analysis (Section II), we *assume* that the temperatures in the drop at *any* time $t > 0$ can be represented by an exponential function of distance from the surface. Putting this in nondimensional form, we have

$$\frac{T - T_i}{T_v - T_i} = \exp\left[\frac{-(r_s - r)}{\delta(t)}\right] \tag{39}$$

where $\delta(t)$ is an arbitrary function, to be determined. Taking $\partial T/\partial r$ in Eq. (39) gives

$$\frac{\partial T}{\partial r} = \frac{(T_v - T_i)}{\delta(t)} \exp\left[\frac{-(r_s - r)}{\delta(t)}\right] \tag{40}$$

But we wish this slope to satisfy the surface boundary condition given by Eq. (27d); thus, using the equation above, evaluated at $r = r_s$, we specify

$$k\left(\frac{\partial T}{\partial r}\right)_{r_s,t} = k\frac{(T_v - T_i)}{\delta(t)} = Q + \rho L\frac{dr_s}{dt}$$

or

$$\delta(t) = \frac{k(T_v - T_i)}{Q + \rho L\dfrac{dr_s}{dt}}$$

so that Eq. (39) becomes

$$\frac{T - T_i}{T_v - T_i} = \exp\left\{-\left[\frac{Q + \rho L\dfrac{dr_s}{dt}}{k(T_v - T_i)}\right](r_s - r)\right\} \tag{41}$$

"This temperature profile satisfies Eq. (29), (30), and (31) approximately. A plot of this and the exact profile obtained from the heatup solution [of Section II], evaluated at a given time [$t = 0$] is shown in Fig. 4.

We define the shorthand quantity

$$\zeta \equiv \left[\frac{Q + \rho L\dfrac{dr_s}{dt}}{k(T_v - T_i)}\right] r_s \tag{42}$$

so that Eq. (41) becomes

$$\frac{T - T_i}{T_v - T_i} = \exp\left(-\zeta + \zeta\frac{r}{r_s}\right)$$

or

$$T = e^{-\zeta + \zeta\frac{r}{r_s}}(T_v - T_i) + T_i \tag{43}$$

In this notation, $\left(\dfrac{\partial T}{\partial r}\right)_{r_s,t}$ is found to be

$$\left(\frac{\partial T}{\partial r}\right)_{r_s,t} = \frac{\zeta}{r_s}(T_v - T_i) \tag{44}$$

At this time it is convenient to differentiate Eq. (42) for future reference.

$$\frac{d\zeta}{dt} = \frac{Q\dfrac{dr_s}{dt} + r_s\dfrac{dQ}{dt}}{k(T_v - T_i)} + \frac{\rho L}{k(T_v - T_i)}\left[\left(\frac{dr_s}{dt}\right)^2 + r_s\frac{d^2r_s}{dt^2}\right]$$

Rearranging, we have

$$\frac{d\zeta}{dt} = \frac{\left[Q + \rho L\left(\dfrac{dr_s}{dt}\right)\right]\dfrac{dr_s}{dt}}{k(T_v - T_i)} + \frac{r_s\left(\dfrac{dQ}{dt} + \rho L\dfrac{d^2r_s}{dt^2}\right)}{k(T_v - T_i)}$$

$$= \frac{\zeta}{r_s}\frac{dr_s}{dt} + \frac{r_s\left(\dfrac{dQ}{dt} + \rho L\dfrac{d^2r_s}{dt^2}\right)}{k(T_v - T_i)} \tag{45}$$

---

2. Initial-Value Problem. Now, we substitute the assumed profile (Eq. 43) into Eq. (38), the heat-balance integral. For the moment, we shall treat only the first term on the left-hand side of Eq. (38), denoting it by F.

$$F \equiv \frac{d}{dt}\int_0^{r_s} Tr\,dr = \frac{d}{dt}\left[\int_0^{r_s} e^{-\zeta + \zeta\frac{r}{r_s}}(T_v - T_i)\,r\,dr + \int_0^{r_s} T_i r\,dr\right]$$

The first integral on the right-hand side can be found in tables. (See Ref. 8.)

$$F = \frac{d}{dt}\left\{\left[(T_v - T_i)\,e^{-\zeta}\left(\frac{r_s\,r\,e^{\zeta\frac{r}{r_s}}}{\zeta} - \frac{r_s^2}{\zeta^2}e^{\zeta\frac{r}{r_s}}\right)\right]_0^{r_s} + \left[\frac{1}{2}T_i\,r^2\right]_0^{r_s}\right\}$$

Insertion of the limits gives

$$F = \frac{d}{dt}\left\{\left[(T_v - T_i)\, e^{-\zeta}\left(\frac{r_s^2 e^{\zeta}}{\zeta} - \frac{r_s^2}{\zeta^2}\, e^{\zeta} + \frac{r_s^2}{\zeta^2}\right)\right] + \frac{1}{2}\, T_i\, r_s^2\right\}$$

$$= \frac{d}{dt}\left\{\left[(T_v - T_i)\, \frac{r_s^2}{\zeta}\left(1 - \frac{1}{\zeta} + \frac{e^{-\zeta}}{\zeta}\right)\right] + \frac{1}{2}\, T_i\, r_s^2\right\}$$

Now we perform the indicated differentiation, by parts:

$$F = (T_v - T_i)\left[\frac{r_s^2}{\zeta}\left(\frac{1}{\zeta^2} - \frac{e^{-\zeta}}{\zeta} - \frac{e^{-\zeta}}{\zeta^2}\right)\frac{d\zeta}{dt} + \left(1 - \frac{1}{\zeta} + \frac{e^{-\zeta}}{\zeta}\right)\left(\frac{2r_s}{\zeta}\frac{dr_s}{dt} - \frac{r_s^2}{\zeta^2}\frac{d\zeta}{dt}\right)\right] + T_i\, r_s\, \frac{dr_s}{dt}$$

collecting derivatives of ζ, we obtain

$$F = (T_v - T_i)\left[\frac{r_s^2}{\zeta^3}\left(1 - \zeta\, e^{-\zeta} - e^{-\zeta} - \zeta + 1 - e^{-\zeta}\right)\frac{d\zeta}{dt} + \left(\zeta - 1 + e^{-\zeta}\right)\frac{2r_s}{\zeta^2}\,\frac{dr_s}{dt}\right] + T_i\, r_s\, \frac{dr_s}{dt}$$

or, in slightly different form,

$$= (T_v - T_i)\left\{\frac{r_s^2}{\zeta^3}\left[2\left(1 - e^{-\zeta}\right) - \zeta\left(1 + e^{-\zeta}\right)\right]\frac{d\zeta}{dt} + \left[\zeta - \left(1 - e^{-\zeta}\right)\right]\frac{2r_s}{\zeta^2}\,\frac{dr_s}{dt}\right\} + T_i\, r_s\, \frac{dr_s}{dt}$$

We shall represent the terms in square brackets by $[B]$ and $[C]$, so that the above equation is

$$F = (T_v - T_i)\left\{\frac{r_s^2}{\zeta^3}[B]\,\frac{d\zeta}{dt} + [C]\,\frac{2r_s}{\zeta^2}\,\frac{dr_s}{dt}\right\} + T_i\, r_s\, \frac{dr_s}{dt} \tag{46}$$

It is useful to relate B to C.

$$B \equiv 2\left(1 - e^{-\zeta}\right) - \zeta\left(1 + e^{-\zeta}\right) \tag{47}$$

Removing parentheses and rearranging, we have

$$B = -\zeta + 2\left(1 - e^{-\zeta}\right) - \zeta\, e^{-\zeta}$$

Now we insert compensating terms:

$$B = \zeta^2 - 2\zeta + 2\left(1 - e^{-\zeta}\right) - \zeta^2 + \zeta - \zeta\, e^{-\zeta}$$
$$= \zeta^2 - 2\left[\zeta - \left(1 - e^{-\zeta}\right)\right] - \zeta\left[\zeta - \left(1 - e^{-\zeta}\right)\right]$$

But the quantity in brackets is C, as implied by Eq. (46), so that

$$B = \zeta^2 - \zeta C - 2C \tag{48}$$

Recalling that F was the first term in Eq. (38), we now retrieve that equation in order to consider the other terms.

$$F - r_s T(r_s, t)\frac{dr_s}{dt} = \alpha\left[r_s\left(\frac{\partial T}{\partial r}\right)_{r_s,t} + T(r_s,t) - T(0,t)\right] \tag{38a}$$

Substitution using Eq. (46), (29), (44), and (43) gives immediately

$$(T_v - T_i)\left(\frac{r_s^2}{\zeta^3} B\frac{d\zeta}{dt} + C\frac{2r_s}{\zeta^2}\frac{dr_s}{dt}\right) + r_s T_i\frac{dr_s}{dt} - r_s T_v\frac{dr_s}{dt} = \alpha\left[\zeta(T_v - T_i) + T_v - e^{-\zeta}(T_v - T_i) - T_i\right]$$

Now we transpose several terms, multiply both sides by ζ^3/r_s^2, and factor out $(T_v - T_i)$.

$$(T_v - T_i) B\frac{d\zeta}{dt} = (T_v - T_i)\left[\frac{\zeta^3\alpha}{r_s^2}(\zeta + 1 - e^{-\zeta}) - \frac{2C\zeta}{r_s}\frac{dr_s}{dt} + \frac{\zeta^3}{r_s}\frac{dr_s}{dt}\right]$$

The derivative of ζ was given by Eq. (45). Using this and transposing two of its terms, we have

$$B\frac{r_s\rho L}{k}\frac{d^2r_s}{dt^2} = (T_v - T_i)\left[\frac{\zeta^3\alpha}{r_s^2}(\zeta + 1 - e^{-\zeta}) - \left(\frac{2C\zeta}{r_s} - \frac{\zeta^3}{r_s} + B\frac{\zeta}{r_s}\right)\frac{dr_s}{dt}\right] - \frac{r_s B\dfrac{dQ}{dt}}{k}$$

Both sides will now be multiplied by $k/(Br_s\rho L)$. At the same time, the expression for B is recalled from Eq. (48).

$$\frac{d^2r_s}{dt^2} = \frac{(T_v - T_i)}{B}\left\{\frac{k}{\rho L}\frac{\zeta^3}{r_s^3}\alpha(\zeta + 1 - e^{-\zeta}) - \frac{k\zeta}{\rho L r_s^2}\left[2C - \zeta^2 + (\zeta^2 - \zeta C - 2C)\right]\frac{dr_s}{dt}\right\} - \frac{1}{\rho L}\frac{dQ}{dt}$$

The right-hand side can be simplified:

$$= (T_v - T_i)\frac{k}{B\rho L}\left(\frac{\zeta}{r_s}\right)^3\left\{\alpha\left[\zeta + (1 - e^{-\zeta})\right] + \left(\frac{r_s}{\zeta^2}\right)\zeta C\right]\frac{dr_s}{dt}\right\} - \frac{1}{\rho L}\frac{dQ}{dt}$$

Now we replace dQ/dt by the expression found in Eq. (27e). We also bring back the definitions of B and C as implied by Eq. (46).

$$\frac{d^2r_s}{dt^2} = \frac{(T_v - T_i)\,k}{\left[2(1 - e^{-\zeta}) - \zeta(1 + e^{-\zeta})\right]\rho L}\left(\frac{\zeta}{r_s}\right)^3\left\{\alpha\left[\zeta + (1 - e^{-\zeta})\right] + \frac{r_s}{\zeta}\left[\zeta - (1 - e^{-\zeta})\right]\frac{dr_s}{dt}\right\} + \frac{4q}{\rho L t^*}e^{-\frac{4t}{t^*}} \tag{49}$$

Thus far, by using the heat balance integral and assuming an exponential form for the drop's temperature profile, we have been able to avoid determining the exact profile in the drop, and obtain the above nonlinear, *ordinary* differential equation which can be solved numerically for $r_s(t)$ if the initial value of r_s is known. (The second integration constant comes from our assumption that vaporization begins at time zero.)

We now make the assumption that ζ is sufficiently large, such that $e^{-\zeta} << 1$. Study of Eq. (43) will show that this assumption implies that the heated zone does not extend to the center of the drop. Most of the time, this may be

reasonable; for, although the drops to be considered are small (diameters of the order of 0.002–0.020 in.), the rate of heat input Q is so great ($\sim$ 40 Btu/in.^{2}/sec) that most of the heat must be used at the surface to vaporize the liquid. The surface regresses at rates of the order of ¼ in./sec, so that there is not time for much heat to be conducted through the liquid. Presumably, then, the center of the drop is not aware of impending doom until a very short time before its occurrence. By this time, the mass of the drop, being proportional to the cube of the radius, may be small enough to neglect. If not, Eq. (49) can be used as it stands; however, if the simplification is allowed, Eq. (49) becomes

$$\frac{d^2r_s}{dt^2} = \frac{(T_v - T_i)\,k}{(2-\zeta)\,\rho L}\left(\frac{\zeta^3}{r_s^3}\right)\left[\alpha(\zeta+1) + \frac{r_s}{\zeta}(\zeta-1)\frac{dr_s}{dt}\right] + \frac{4\bar{q}}{\rho L t^*}e^{-\frac{4t}{t^*}} \tag{50}$$

3. Nondimensional form. Equation (50) can be of more general use if it is expressed in terms of nondimensional combinations of the physical parameters involved. To begin with, we define a nondimensional time:

$$\tau \equiv \frac{\alpha t}{a^2} \tag{51}$$

and a drop radius related to the initial radius:

$$\eta \equiv \frac{r_s}{a} \tag{52}$$

The τ-derivative of this quantity will be denoted by $\dot{\eta}$.

$$\dot{\eta} \equiv \frac{d\eta}{d\tau} = \frac{d\eta}{dt}\frac{dt}{d\tau} = \left[\frac{d\,\frac{r_s}{a}}{dt}\right]\frac{a^2}{\alpha} = \frac{a}{\alpha}\frac{dr_s}{dt} \tag{53}$$

$$\ddot{\eta} \equiv \frac{d^2\eta}{d\tau^2} = \frac{d\dot{\eta}}{dt}\frac{dt}{d\tau} = \left(\frac{a}{\alpha}\frac{d^2r_s}{dt^2}\right)\frac{a^2}{\alpha} = \frac{a^3}{\alpha^2}\frac{d^2r_s}{dt^2} \tag{54}$$

The last expression prompts us to multiply Eq. (50) by a^3/α^2; at the same time, we substitute the definition of ζ from Eq. (42), as well as introducing mutually compensating sign changes. Eq. (50) becomes

$$\ddot{\eta} = -\frac{a^3}{\alpha^2}\frac{k(T_v - T_i)}{\rho L}\frac{1}{\left[\frac{\left(Q+\rho L\frac{dr_s}{dt}\right)r_s}{k(T_v-T_i)} - 2\right]}\left[\frac{\left(Q+\rho L\frac{dr_s}{dt}\right)^3}{k^3(T_v-T_i)^3}\right]\left\{\alpha\left[\frac{\left(Q+\rho L\frac{dr_s}{dt}\right)r_s}{k(T_v-T_i)} + 1\right]\right.$$

$$\left. + \frac{k(T_v-T_i)}{\left(Q+\rho L\frac{dr_s}{dt}\right)}\left[\frac{\left(Q+\rho L\frac{dr_s}{dt}\right)r_s}{k(T_v-T_i)} - 1\right]\frac{dr_s}{dt}\right\} + \frac{4\bar{q}a^3}{\alpha^2\rho L t^*}e^{-4t/t^*}$$

The right-hand side is now multiplied and divided by $a/(\rho L\alpha)$ several times in order to give nondimensional terms. Several other terms are rearranged for the same purpose.

$$\ddot{\eta} = -\frac{a^3}{\alpha}\left[\frac{k(T_v - T_i)}{\alpha\rho L}\right]\frac{\dfrac{a}{\rho L\alpha}\,k(T_v - T_i)}{\left[\left(\dfrac{aQ}{\rho L\alpha} + \dfrac{a}{\alpha}\dfrac{dr_s}{dt}\right)r_s - \dfrac{2ak}{\rho L\alpha}(T_v - T_i)\right]}\left\{\frac{\left(\dfrac{aQ}{\rho L\alpha} + \dfrac{a}{\alpha}\dfrac{dr_s}{dt}\right)^3}{a^3\left[\dfrac{k(T_v - T_i)}{\rho L\alpha}\right]^3}\right\}\left\{\alpha\left[\frac{\left(\dfrac{aQ}{\rho L\alpha} + \dfrac{a}{\alpha}\dfrac{dr_s}{dt}\right)\dfrac{r_s}{a}}{\dfrac{k(T_v - T_i)}{\rho L\alpha}} + 1\right]\right.$$

$$\left. + \frac{\dfrac{a}{\rho L\alpha}\,k(T_v - T_i)}{\dfrac{aQ}{\rho L\alpha} + \dfrac{a}{\alpha}\dfrac{dr_s}{dt}}\left[\frac{\left(\dfrac{aQ}{\rho L\alpha} + \dfrac{a}{\alpha}\dfrac{dr_s}{dt}\right)\dfrac{r_s}{a}}{\dfrac{k(T_v - T_i)}{\rho L\alpha}} - 1\right]\frac{dr_s}{dt}\right\} + \frac{4a^3\bar{q}}{\rho L\alpha^2 t^*}\exp\left[-\left(\frac{4a^2}{\alpha t^*}\right)\left(\frac{\alpha t}{a^2}\right)\right] \tag{55}$$

Nondimensional parameters will now be defined:

$$\Lambda \equiv \frac{k(T_v - T_i)}{\rho L\alpha} \qquad \gamma \equiv \frac{ah(T_f - T_v)}{\rho L\alpha} \qquad \beta \equiv \frac{4a^2}{\alpha t^*}$$

$$\bar{\gamma} \equiv \frac{\bar{q}a}{\rho L\alpha} \qquad \psi \equiv \bar{\gamma}\,e^{-\beta\tau} + \gamma + \dot{\eta} = \frac{a}{\rho L\alpha}\left[\bar{q}\,e^{-4t/t^*} + h(T_f - T_v) + \rho L\frac{dr_s}{dt}\right] = \frac{aQ}{\rho L\alpha} + \frac{a}{\alpha}\frac{dr_s}{dt} \tag{56}$$

and we recall that $\eta \equiv r_s/a$, $\tau \equiv \alpha t/a^2$, so that Eq. (55) becomes (cancelling a^3/α):

$$\ddot{\eta} = -[\Lambda]\frac{a\Lambda}{[\psi\eta a - 2a\Lambda]}\left\{\frac{\psi^3}{\Lambda^3}\right\}\left\{\left[\frac{\psi\eta}{\Lambda} + 1\right] + \left(\frac{1}{\alpha}\right)\frac{a\Lambda}{\psi}\left[\frac{\psi\eta}{\Lambda} - 1\right]\frac{dr_s}{dt}\right\} + \bar{\gamma}\,\beta\,e^{-\beta\tau}$$

A (ψ) will now be moved inside the latter set of braces above, and the term $1/\Lambda$ factored out. Some other terms cancel, and we use

$$\dot{\eta} = \frac{a}{\alpha}\frac{dr_s}{dt}$$

found in Eq. (53).

$$\ddot{\eta} = \frac{-\psi^2}{(\psi\eta - 2\Lambda)\Lambda^2}\left\{[\psi^2\eta + \Lambda\psi] + [\Lambda\psi\,\eta\,\dot{\eta} - \Lambda^2\,\dot{\eta}]\right\} + \bar{\gamma}\,\beta\,e^{-\beta\tau}$$

A slight rearrangement gives

$$\boxed{\ddot{\eta} = \frac{-\psi^2}{(\psi\eta - 2\Lambda)\Lambda^2}\left[\psi^2\eta + \Lambda\psi + \Lambda\dot{\eta}\,(\psi\eta - \Lambda)\right] + \bar{\gamma}\,\beta\,e^{-\beta\tau}} \tag{57}$$

A simpler form of this equation is possible for the case of a steady heat flux, or $\bar{q} = 0$. Reference to Eq. (56) shows that this implies $\bar{\gamma} = 0$, and $\psi = \gamma + \dot{\eta}$, so that Eq. (57) becomes

$$\ddot{\eta} = \frac{-(\gamma + \dot{\eta})^2}{[\eta(\gamma + \dot{\eta}) - 2\Lambda]\Lambda^2}\left[(\gamma + \dot{\eta})^2\eta + \Lambda(\gamma + \dot{\eta}) + \Lambda\,\dot{\eta}\,(\gamma\,\eta + \dot{\eta}\,\eta - \Lambda)\right] \tag{58}$$

We can obtain a useful *approximate* solution to this equation by assuming that $\dot{\eta} = -\gamma$. This amounts to an assumption that the heat used to raise the liquid temperature is negligible compared with the amount that goes into the vapors as latent and sensible heat. Calculation will show us whether or not this is a valid assumption. We also note that, as γ is a constant, the assumption implies that $\ddot{\eta} = 0$, or that the surface regresses at a uniform rate. We shall call this rate $\dot{\eta}_{ss}$.

Integration of $\dot{\eta}_{ss} = -\gamma$ gives

$$\eta_{ss} = -\gamma\tau + C$$

where C is the constant of integration. By definition, the value of the nondimensional radius at $t = \tau = 0$ is unity, which means that $C = 1$. Therefore,

$$\eta_{ss} = 1 - \gamma\tau \tag{59}$$

C. Application to an Engine Environment

1. Steady heat input. "The results of this analysis are applied to the determination of the vaporization rate of a liquid fuel droplet moving through a rocket motor. The chamber is divided into a number of sections where the thermodynamic properties are specified and assumed constant for the interval. Thus, once the variation of film coefficient, flame temperature, and mean droplet velocity with chamber location are known, the burning rate and droplet history can be determined using the preceding analysis. The steady-state solution, Eq. (59), is used as an approximation, or the more exact Eq. (58) can be integrated numerically." For the latter process, we may obtain values of η at time $\tau_{(i+1)}$ by expanding η in a Taylor's series to second order about the point τ_i.

$$\eta(\tau_{i+1}) \doteq \eta(\tau_i) + \dot{\eta}(\tau_i)\Delta\tau + \ddot{\eta}(\tau_i)\frac{(\Delta\tau)^2}{2!} \tag{60}$$

Differentiation of this with respect to τ gives

$$\dot{\eta}(\tau_{i+1}) \doteq \dot{\eta}(\tau_i) + \ddot{\eta}(\tau_i)\,\Delta\tau \tag{61}$$

"The values of $\eta(\tau_i)$ and $\dot{\eta}(\tau_1)$ can be substituted into Eq. (58) to obtain $\ddot{\eta}(\tau_i)$. Initially, $\eta(0) = 1$ and $\dot{\eta}(0) = 0$.

"The steady-state and transient vaporization rate solutions are applied step-wise to a particular rocket motor for the purpose of comparison. The steady-state vaporization rate (Eq. 59) is found to be in agreement with the transient evaporation rate (Eq. 58) and for the particular fuel considered, the variation is within 2%. The droplet vaporization rate is shown as a function of chamber location [for a typical set of parameters that were not specified by Grossman] in Fig. 6. For the remainder of this discussion, the steady-state solution to Eq. (58) is used.

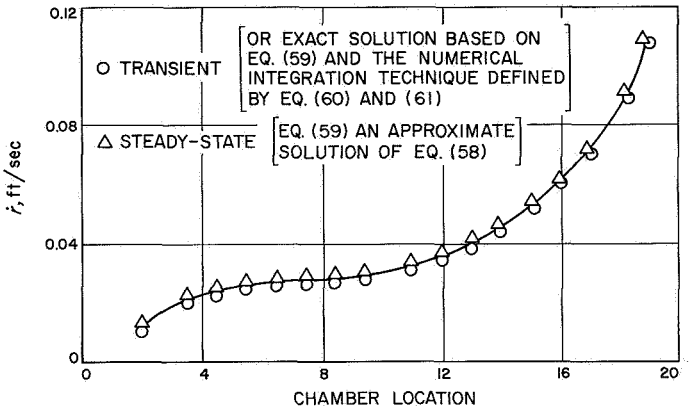

Fig. 6. Transient and steady-state vaporization rates

"The vaporization rate of a fuel droplet in a rocket motor determined in this paper is compared with the vaporization rate calculated from the method given in Burstein, et al. (Ref. 9).[6] The ballistics of 50/50 Aerozine-nitrogen tetroxide propellants are determined. The thermodynamic properties calculated in Burstein's analysis are used here and a plot of some of these is shown in Fig. 7 and 8. For the propellant combination considered, a modification of the energy balance, Eq. (27a) is made. Since the dissociation of hydrazine is exothermic, a heat of reaction q_r must be included in the heat source term. Thus Eq. (27a) becomes

$$h(T_f - T_i) = k\left(\frac{\partial T}{\partial r}\right)_{r_s,t} - \rho[\lambda + c_p(T_f - T_v) + q_r]\frac{dr_s}{dt} \tag{62}$$

[Assuming a constant q_r, the derivations are not changed if L (Eq. 27c) is redefined to include q_r.]

"Now with these modifications, the two methods are used to calculate the variation of the vaporization rate

[6] The Burstein model assumes "rapid circulation" within the drop, so that the surface temperature is at all times equal to the average drop temperature.

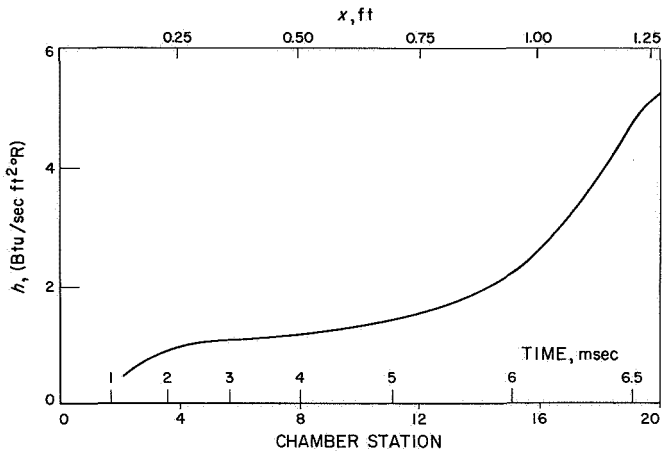

Fig. 7. Film coefficient 50/50 Aerozine

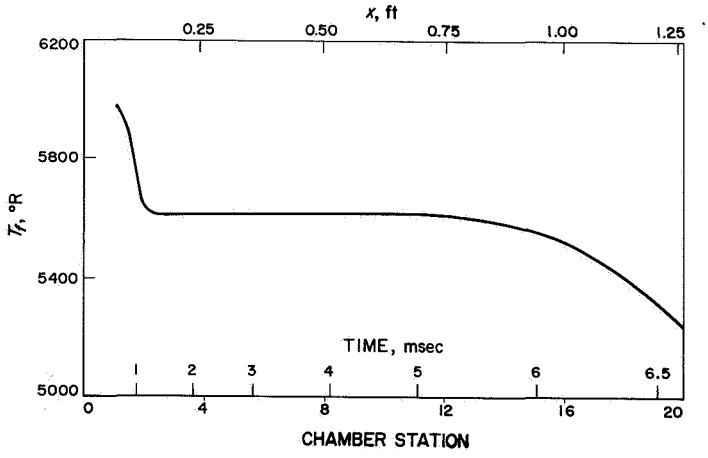

Fig. 8. Flame temperature 50/50 Aerozine

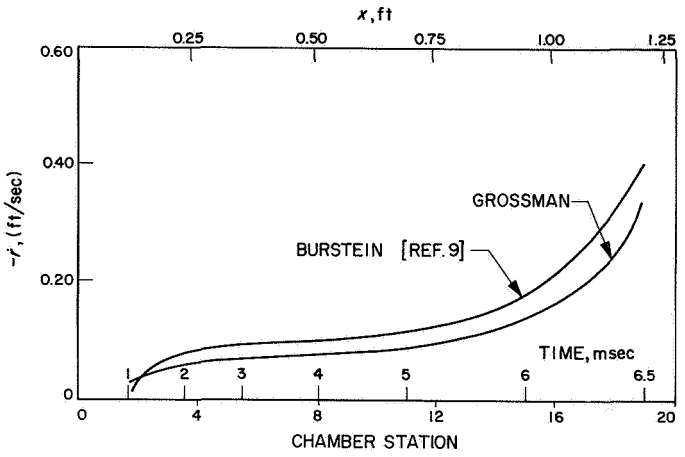

Fig. 9. Vaporization rate vs chamber location 50/50 Aerozine

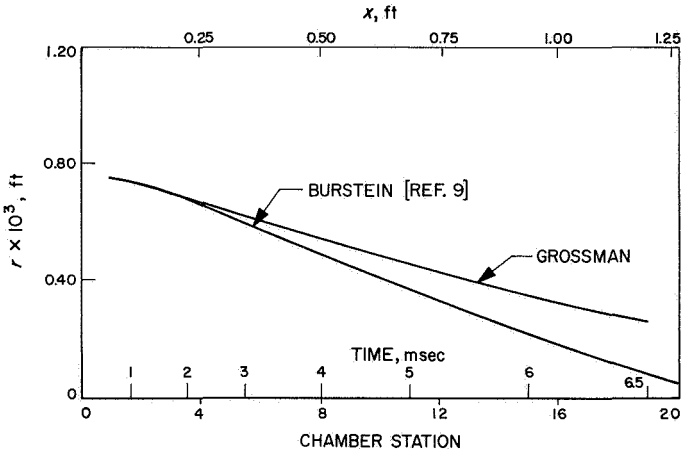

Fig. 10. Droplet radius vs chamber location 50/50 Aerozine

and droplet radius with chamber location. The results of the two analyses are given in Fig. 9 and 10. The technique given in this paper [yields results that are] within 20% of [those obtained by the method described] in Ref. 9.

2. Nonsteady heat input. "... Equation (57) possesses similar characteristics to Eq. (58) and may be solved using the same numerical integration, Eq. (60) and (61).

"This solution is now applied step-wise to the rocket motor which has been previously described. The unsteady heat input is applied [once] at each chamber section and hence the effect of chamber location on the additional burning rate is found. Mass conservation, Eq. (28) is used to yield the additional mass evaporation due to the unsteady heat input. For a $\bar{q}$ of 20% [of $h(T_f - T_v)$] and time constants $t^* = 1$, 0.1, and 0.01 millisec, the [total] additional mass evaporation with chamber section is calculated and plotted in Fig. 11."

The effects of larger heat pulses have been calculated at JPL by Dr. R. Kushida. In order to separate the heat-up transient from the exponential heat pulse, $\bar{q}$ was initially set to zero. The reference diameter, a, was arbitrarily redefined to be 225 μ. A 315-μ droplet was considered, so that the initial value of nondimensional diameter η was 1.4. When evaporation had reduced the drop's radius to 225 μ ($\eta = 1.0$), the heat input was increased by a factor of six, then allowed to decay exponentially; that is [referring to Eq. (27)],

$$\bar{q} = 5\,h(T_f - T_v)$$

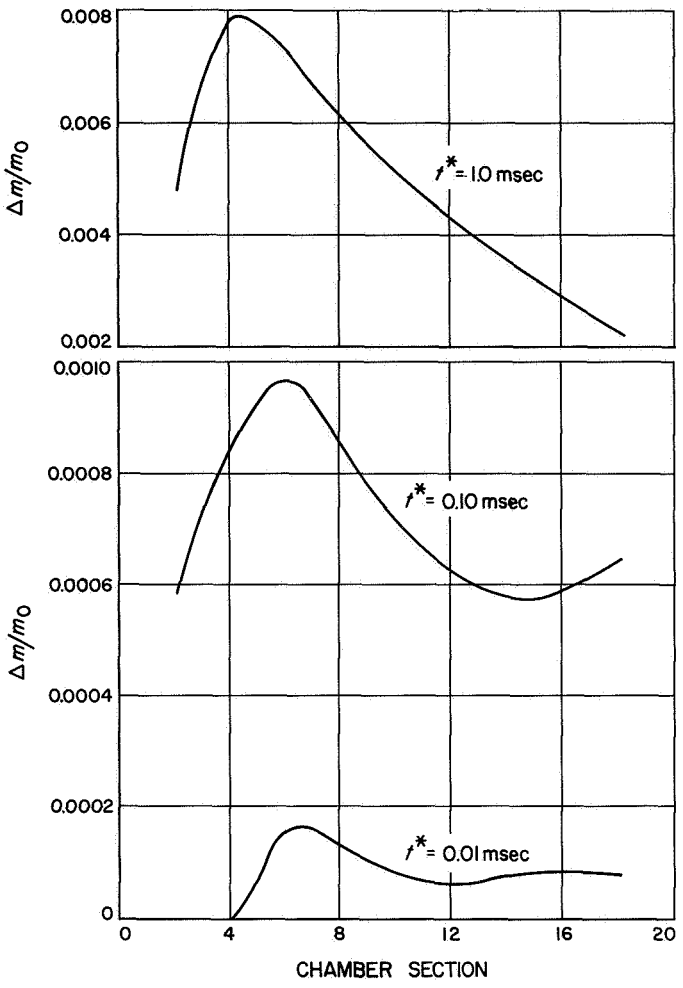

Fig. 11. Additional mass release due to unsteady heat input

with a decay time $t^* = 0.1$ millisec. A pulse of this magnitude and duration is of interest in severe cases of nonlinear combustion instability.

Typical results are shown in Fig. 12 and 13. The low-frequency (heat-up) history is shown in Fig. 12, for a "pre-heated" drop with uniform initial temperature $T_i = 300°F$. Figure 13 shows the effect of the high-frequency heat pulse. Vaporization rate follows closely the exponential form of the heat flux, and the percent of burning rate increase is almost exactly equal to the percent increase in heat input; the relatively high temperature of this drop's interior causes conduction to be of negligible influence during the heat pulse. (Even for much cooler drops, however, vaporization response was found to be very strong.)

Table 1 gives the parameters used for the case shown in Fig. 12 and 13.

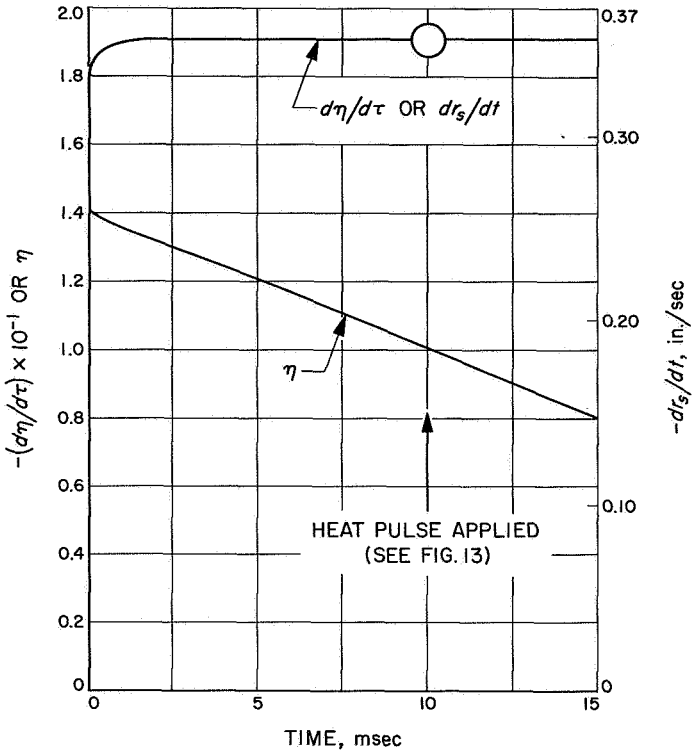

Fig. 12. Heatup and evaporation of a drop

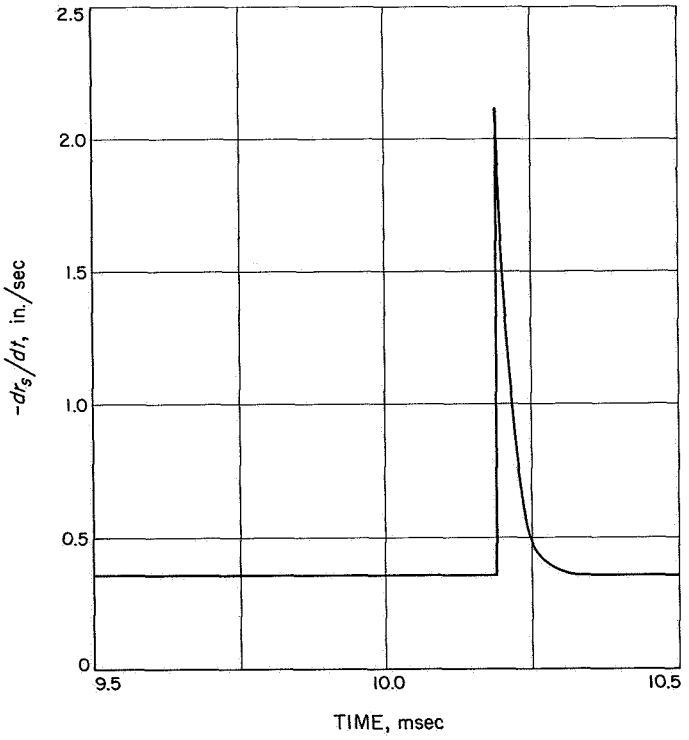

Fig. 13. Response of a drop to a high-frequency heat pulse

Table 1. Parameters used in Fig. 12 and 13

$T_i = 760°R$
$L = 1500$ Btu/lbm
$\rho = 53$ lbm/ft^3
$h = 0.5$ Btu/sec ft^2 °R
$\alpha = 1.15 \times 10^{-6}$ ft^2/sec
$T_f = 5630°R$
$T_v = 860°R$
$t^* = 0.1$ millisec

D. Conclusion

"A heat transfer analysis for the determination of the vaporization rate of a liquid fuel droplet has been presented. Steady-state and transient vaporization rate solutions were obtained. The vaporization rate given herein was compared to that calculated using Burstein's (Ref. 9) analysis which is based on a diffusion limited model. Both solutions were found to be in agreement and the vaporization rates differed by a maximum of 20%.

"It has also been shown that an amount of additional mass can be released due to the passage of a wave over an evaporating droplet and that favorable positions in a rocket chamber exist which support this mass release.

"Finally, it may be concluded that a working analysis which can determine the propellant vaporization rates of a liquid fuel droplet with steady and unsteady heat inputs has been presented."

Nomenclature

a	initial droplet radius, ft
A	surface area of the droplet, ft^2
c_p	specific heat, at constant pressure, of vapors, Btu/lb°F
C_1, C_2	constants of integration, ft sec°R
C_p	specific heat, Btu/lbm°R
h	film heat-transfer coefficient, Btu/ft^2 sec°R
k	thermal conductivity, Btu/sec ft°R
L	heat to vaporize the liquid and raise vapor to the flame temperature, Btu/lbm
m_0	unperturbed mass flow, lbm
q	Laplace variable, sec^{-1}
$\bar{q}$	peak additional heat input, Btu/ft^2 sec
q_r	heat of reaction, Btu/lbm
q_t	total heat transfer from the flame, Btu/ft^2 sec
Q	rate of heat flow into the film, Btu/ft^2 sec
r	radial coordinate, ft
r'	nondimensional radius
r_s	radius of the vaporizing droplet, ft
t	time, sec
T	temperature, °R
$\bar{T}_d$	mean droplet temperature, °R
T_f	flame temperature, °R
T_i	initial or injection temperature, °R
T_v	vaporization temperature, °R
u	transformed variable, rT or $r(T_f - T)$, ft °R
$\bar{u}$	Laplace transform of u, ft sec °R
w	mass vaporization rate, lbm/sec
α	thermal diffusivity, ft^2 sec
β	defined in Eq. (56)
γ	heat-transfer parameter defined in Eq. (56)
$\bar{\gamma}$	nonsteady heat-transfer parameter defined in Eq. (56)
δ	arbitrary function of time, ft
η	droplet radius
$\dot{\eta}$	derivative of η with respect to nondimensional time
λ	heat of vaporization
Λ	defined in Eq. (56), Btu/lb

Nomenclature (contd)

ζ defined in Eq. (42)

μ $\left(\frac{ah}{k} - 1\right)$

ρ density of the liquid, lbm/ft^3

τ time

ψ defined in Eq. (56)

Subscripts

a droplet surface

c complementary

f stagnation of flow field

i initial

p particular

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